

Generalizations of Quandle Cocycle Invariants and Alexander Modules from Quandle Modules

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1 Introduction

Quandle cohomology theory was developed [5] to define invariants of classical knots and knotted surfaces in state-sum form, called quandle cocycle (knot) invariants. The quandle cohomology theory is a modification of rack cohomology theory which was defined in [11]. The cocycle knot invariants are analogous in their definitions to the Dijkgraaf-Witten invariants [8] of triangulated 3-manifolds with finite gauge groups, but they use quandle knot colorings as spins and cocycles as Boltzmann weights. In [4], the quandle cocycle invariants were generalized in three different directions, using generalizations of quandle homology theory provided by Andraskiewicz and Graña [1], which is compared to the group cohomology theories with the group actions on the coefficient groups. This paper is a written version of our talk given at Intelligence of Low Dimensional Topology in Shodo-Shima. It is a short summary of [4] with some results from [6] and a few new observations. We would like to thank the organizers for holding such an exciting conference in a beautiful location.

2 Preliminary: Quandles and colorings

A *quandle*, X , is a set with a binary operation $(a, b) \mapsto a * b$ such that

- (I) For any $a \in X$, $a * a = a$.
- (II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.
- (III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

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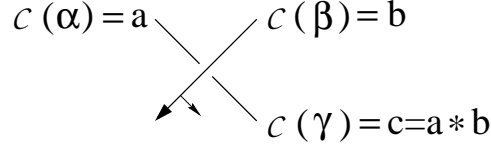


Figure 1: Quandle relation at a crossing

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [1, 3, 10, 18, 20, 30].

The following are typical examples of quandles. A group $X = G$ with conjugation as the quandle operation: $a * b = bab^{-1}$. We denote by $\text{Conj}(G)$ the quandle defined for a group G by $a * b = bab^{-1}$. Any subset of G that is closed under such conjugation is also a quandle.

Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$ -module M is a quandle with $a * b = ta + (1 - t)b$, $a, b \in M$, that is called an *Alexander quandle*. Let n be a positive integer, and for elements $i, j \in \{0, 1, \dots, n-1\}$, define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n . This set can be identified with the set of reflections of a regular n -gon with conjugation as the quandle operation; it also is isomorphic to an Alexander quandle $\mathbb{Z}_n[t, t^{-1}]/(t + 1)$. As a set of reflections of the regular n -gon, R_n can be considered as a subquandle of $\text{Conj}(\Sigma_n)$.

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let \mathcal{R} be the set of (over-)arcs. The normals are given in such a way that (tangent, normal) agrees with the orientation of the plane, see Fig. 1. A (quandle) *coloring* \mathcal{C} is a map $\mathcal{C} : \mathcal{R} \rightarrow X$ such that at every crossing, the relation depicted in Fig. 1 holds. More specifically, let β be the over-arc at a crossing, and α, γ be under-arcs such that the normal of the over-arc points from α to γ . (In this case, α is called the *source arc* and γ is called the *target arc*.) Then it is required that $\mathcal{C}(\gamma) = \mathcal{C}(\alpha) * \mathcal{C}(\beta)$. The colors $\mathcal{C}(\alpha), \mathcal{C}(\beta)$ are called *source colors*.

3 Quandle Modules

We recall some information from [1], but with notation changed to match our conventions.

Let X be a quandle. Let $\Omega(X)$ be the free \mathbb{Z} -algebra generated by $\eta_{x,y}, \tau_{x,y}$ for $x, y \in X$ such that $\eta_{x,y}$ is invertible for every $x, y \in X$. Define $\mathbb{Z}(X)$ to be the quotient $\mathbb{Z}(X) = \Omega(X)/R$ where R is the ideal generated by

$$1. \quad \eta_{x*y,z}\eta_{x,y} - \eta_{x*z,y*z}\eta_{x,z}$$

2. $\eta_{x*y,z}\tau_{x,y} - \tau_{x*z,y*z}\eta_{y,z}$
3. $\tau_{x*y,z} - \eta_{x*z,y*z}\tau_{x,z} - \tau_{x*z,y*z}\tau_{y,z}$
4. $\tau_{x,x} + \eta_{x,x} - 1$

The algebra $\mathbb{Z}(X)$ thus defined is called the *quandle algebra* over X . In $\mathbb{Z}(X)$, we define elements $\overline{\eta}_{z,y} = \eta_{z*y,y}^{-1}$ and $\overline{\tau}_{z,y} = -\overline{\eta}_{z,y}\tau_{z*y,y}$.

A *representation* of $\mathbb{Z}(X)$ is an algebra homomorphism $\mathbb{Z}(X) \rightarrow \text{End}(G)$, and we denote the image of the generators by the same symbols. Given a representation of $\mathbb{Z}(X)$ we say that G is a $\mathbb{Z}(X)$ -module, or a *quandle module*. The action of $\mathbb{Z}(X)$ on G is written by the left action, and denoted by $(\rho, g) \mapsto \rho g (= \rho \cdot g = \rho(g))$, for $g \in G$ and $\rho \in \text{End}(G)$.

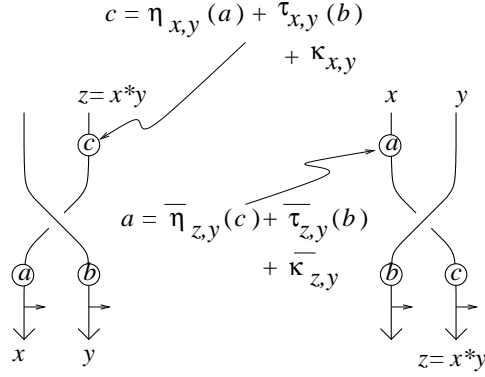


Figure 2: The geometric notation at a crossing

Diagrammatic conventions of η and τ are depicted in Fig. 2, where κ is a generalized 2-cocycle that will appear later in this paper. We have to leave details to [4].

Example 3.1 [1] Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ denote the ring of Laurent polynomials. Then any Λ -module M is a $\mathbb{Z}(X)$ -module for any quandle X , by $\eta_{x,y}(a) = ta$ and $\tau_{x,y}(b) = (1 - t)(b)$ for any $x, y \in X$.

The group $G_X = \langle x \in X \mid x * y = yxy^{-1} \rangle$ is called the *enveloping group* [1] (and the *associated group* in [10]). For any quandle X , any G_X -module M is a $\mathbb{Z}(X)$ -module by $\eta_{x,y}(a) = ya$ and $\tau_{x,y}(b) = (1 - x * y)(b)$, where $x, y \in X$, $a, b \in M$.

Remark 3.2 The second example above is the Fox's free derivative of the braid group representation into the automorphism group of the free group which corresponds to $(x, y) \mapsto (y, yxy^{-1})$. Indeed, one computes

$$\eta_{x,y} = \frac{\partial}{\partial x}(yxy^{-1}) = y, \quad \tau_{x,y} = \frac{\partial}{\partial y}(yxy^{-1}) = 1 - yxy^{-1}.$$

Wada listed certain types of braid group representations. Among them are of the form that a standard braid generator acts as $(x, y) \mapsto (y, w(x, y))$, where $w(x, y) = y^m xy^{-m}$ for some integer m , and $w(x, y) = yx^{-1}y$. We remark here that the Fox's derivative of these give rise to quandle module structures as well. Cocycle invariants for these quandle module structures would be of future research interest.

4 Generalized quandle homology theory

Consider the free right $\mathbb{Z}(X)$ -module $C_n(X) = \mathbb{Z}(X)X^n$ with basis X^n (for $n = 0$, X^0 is a singleton $\{x_0\}$, for a fixed element $x_0 \in X$). In [1], boundary operators $\partial = \partial_n : C_{n+1}(X) \rightarrow C_n(X)$ are defined by

$$\begin{aligned} \partial(x_1, \dots, x_{n+1}) &= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \eta_{[x_1, \dots, \widehat{x}_i, \dots, x_{n+1}], [x_i, \dots, x_{n+1}]}(x_1, \dots, \widehat{x}_i, \dots, x_{n+1}) \\ &\quad - (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_{n+1}) \\ &\quad + (-1)^{n+1} \tau_{[x_1, x_3, \dots, x_{n+1}], [x_2, x_3, \dots, x_{n+1}]}(x_2, \dots, x_{n+1}), \end{aligned}$$

where $[x_1, x_2, \dots, x_n] = ((\dots (x_1 * x_2) * x_3) * \dots) * x_n$

for $n > 0$, and $\partial_1(x) = -\tau_{x * x_0, x_0}$ for $n = 0$. The notational conventions are slightly different from [1]. In particular, the 2-cocycle condition for a 2-cochain $\kappa_{x,y}$ in this homology theory is written as

$$\eta_{x*y,z}(\kappa_{x,y}) + \kappa_{x*y,z} = \eta_{x*z,y*z}(\kappa_{x,z}) + \tau_{x*z,y*z}(\kappa_{y,z}) + \kappa_{x*z,y*z},$$

for any $x, y, z \in X$. We call this a *generalized (rack) 2-cocycle condition*. When κ further satisfies $\kappa_{x,x} = 0$ for any $x \in X$, we call it a *generalized quandle 2-cocycle*.

5 Assigning homology classes to colored diagrams

Here we review only the knotted surface case. The classical case is similar and the triple points are replaced by crossings and 3-cocycles are replaced by 2-cocycles. This method was independently developed in [39].

A diagram D of a knotted surface K is given in 3-space. We assume the surface is oriented and use orientation normals to indicate the orientation. In a neighborhood of each triple point, there are eight regions that are separated

by the sheets of the surface since the triple point looks like the intersection of the 3-coordinate planes in some parametrization. The region into which all normals point is called the *target* region. Let γ be an arc from the region at infinity of 3-space to the target region of a given triple point r . Assume that γ intersects D transversely in a finitely many points thereby missing double point curves, branch points, and triple points. Let a_i , $i = 1, \dots, k$, in this order, be the sheets of D that intersect γ from the region at infinity to the triple point r . Let \mathcal{C} be a coloring of D by a fixed finite quandle X . See Fig. 3.

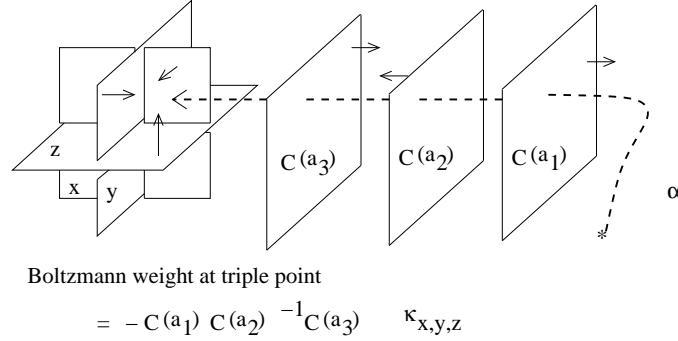


Figure 3: The weight at a triple point

Definition 5.1 The 3-chain

$$\mathcal{C}(D) = \sum_r \epsilon(r) (\mathcal{C}(a_1)^{\epsilon(a_1)} \mathcal{C}(a_2)^{\epsilon(a_2)} \dots \mathcal{C}(a_k)^{\epsilon(a_k)}) (x, y, z) \in C_3(X; \mathbb{Z}G_X)$$

is called the 3-chain represented by the diagram D with the coloring \mathcal{C} .

The following is proved from definitions.

Lemma 5.2 For any knotted surface diagram D with a coloring \mathcal{C} , the 3-chain $\mathcal{C}(D)$ represented by the diagram D with the coloring \mathcal{C} is a 3-cycle: $\mathcal{C}(D) \in Z_3(X; \mathbb{Z}G_X)$.

6 Cocycle invariants

We continue with the surface case, as the classical case is similar.

Let X be a finite quandle and $\mathbb{Z}(X)$ be its quandle algebra with generators $\{\eta_{x,y}^{\pm 1}\}_{x,y \in X}$ and $\{\tau_{x,y}\}_{x,y \in X}$. Let G be an abelian group that is a G_X -module. Recall that this induces a $\mathbb{Z}(X)$ -module structure given by

$\eta_{x,y}g = yg$ and $\tau_{x,y}(g) = (1 - x * y)g$ for $g \in G$ and $x, y \in X$. Let $\kappa_{x,y,z}$ be a generalized quandle 3-cocycle of X with the coefficient group G . Thus the generalized 3-cocycle condition, in this setting, is written as

$$\begin{aligned} & w\kappa_{x,y,z} + \kappa_{x*z,y*z,w} + ((y * z) * w)\kappa_{x,z,w} + \kappa_{y,z,w} \\ &= (((x * y) * z) * w)\kappa_{y,z,w} + \kappa_{x*y,z,w} + (z * w)\kappa_{x,y,w} + \kappa_{x*w,y*w,z*w}. \end{aligned}$$

We require further that $\kappa_{x,x,y} = \kappa_{x,y,y} = 0$. These conditions are called *quandle cocycle conditions*, and a 3-cocycle that satisfies the quandle cocycle conditions is called a (generalized) *quandle 3-cocycle*. A cocycle invariant of knotted surfaces will be defined using such a 3-cocycle.

Definition 6.1 The *Boltzmann weight* $B(\mathcal{C}, r, \gamma)$ for the triple point r , for a coloring \mathcal{C} , with respect to γ , is defined by

$$B(\mathcal{C}, r, \gamma) = \epsilon(r)(\mathcal{C}(a_1)^{\epsilon(a_1)}\mathcal{C}(a_2)^{\epsilon(a_2)} \cdots \mathcal{C}(a_k)^{\epsilon(a_k)})\kappa_{x,y,z} \in G,$$

where x, y, z are the *color triplet* at the given triple point r (x is assigned on the bottom sheet from which the normals of the middle and top sheets point, and y is assigned to the middle sheet from which the normal of the top sheet points, and z is assigned to the top sheet). The sign $\epsilon(r)$ is the sign of the triple point r . The exponent $\epsilon(a_j)$ is 1 if the arc γ crosses the arc a_j against its normal, and is -1 otherwise, for $j = 1, \dots, k$.

Definition 6.2 [4] The family $\Phi_\kappa(K) = \{\sum_r B(\mathcal{C}, r)\}_{\mathcal{C} \in \text{Col}_X(D)}$ is called the quandle cocycle invariant with respect to the (generalized) 3-cocycle κ .

Theorem 6.3 [4] *The family $\Phi_\kappa(K)$ does not depend on the choice of a diagram D of a given knotted surface K , so that it is a well-defined knot invariant.*

The cocycle invariant can be regarded as a family over all colorings (or the formal sum) of the Kronecker product $B(\mathcal{C}, r) = \langle \kappa, \mathcal{C}(D) \rangle$.

7 Computations

The generalized cocycle invariants were computed using *Maple* and *Mathematica* in [4]. Here we include the table of the invariant for a certain 3-cocycle of R_3 with coefficient group \mathbb{Z}^3 where R_3 acts as permutations of factors of the vectors in \mathbb{Z}^3 . The table is for the 2-twist spin of classical knots in the table, up to 8 crossings. Those for up to 9 crossing were computed in [4]. The 3-cocycle used has two free variables q_1 and q_2 , so that the values in the table contain these. The notation \sqcup_n indicates n copies of the vector.

Knot K	$\Phi_\kappa(\text{Tw}^2(K))$
3_1	$\sqcup_9(0, 0, 0)$.
6_1	$\sqcup_3(0, 0, 0), (-q_2, 0, q_2), (q_2, q_1, -q_1 - q_2), (q_1, 0, -q_1),$ $(0, -q_1, q_1), (-q_1 + q_2, q_1 - q_2, 0), (-q_2, -q_1 + q_2, q_1)$.
7_4	$\sqcup_3(0, 0, 0), (-q_2, -2q_1, 2q_1 + q_2), (-q_1 + q_2, q_1, -q_2),$ $(-q_1, 2q_1, -q_1), (-q_1, -q_1, 2q_1), (q_2, -q_2, 0), (-q_2, q_2, 0)$.
7_7	$\sqcup_3(0, 0, 0), \sqcup_2(q_1, 0, -q_1), \sqcup_2(0, -q_1, q_1), \sqcup_2(-q_1, q_1, 0)$.
8_5	$\sqcup_9(0, 0, 0)$.
8_{10}	$\sqcup_9(0, 0, 0)$.
8_{11}	$\sqcup_3(0, 0, 0), (q_2, 0, -q_2), (-q_2, -q_1, q_1 + q_2), (-q_1, 0, q_1),$ $(0, q_1, -q_1), (q_1 - q_2, -q_1 + q_2, 0), (q_2, q_1 - q_2, -q_1)$.
8_{15}	$\sqcup_3(0, 0, 0), \sqcup_3(0, -q_1, q_1), \sqcup_3(-q_1, q_1, 0)$.
8_{18}	$\sqcup_9(0, 0, 0), \sqcup_6(q_1, 0, -q_1), \sqcup_6(-q_1, q_1, 0), \sqcup_6(0, -q_1, q_1)$.
8_{19}	$\sqcup_9(0, 0, 0)$.
8_{20}	$\sqcup_9(0, 0, 0)$.
8_{21}	$\sqcup_9(0, 0, 0)$.

Table 1: A table of cocycle invariants for twist spun knots

8 Applications

The following topological applications have been found.

Non-invertibility of knotted surfaces. A knot(ed surface) is called invertible if it is equivalent to itself with the opposite orientation, with the orientation of the space fixed.

Fox [12] presented a non-invertible knotted sphere using asymmetric knot modules. Farber [9] showed that the 2-twist spun trefoil was non-invertible using the Farber-Levine pairing (see also Hillman [16]). Ruberman [32] used Casson-Gordon invariants to prove the same result, with other new examples of non-invertible knotted spheres. Neither technique applies directly to the same knot with trivial 1-handles attached (in this case the knot is a surface with a higher genus). Kawauchi [21, 22] has generalized the Farber-Levine pairing to higher genus surfaces, showing that such a surface is also non-invertible. Gordon [14] showed that a large family of knotted spheres are indeed non-invertible using fibrations.

In [5] the original quandle cocycle invariants were used to detect non-invertibility of the 2-twist spun trefoil. Furthermore, using cocycle invariants for proving non-invertibility applies to stabilized surfaces. These are obtained by attaching trivial 1-handles. Satoh [2] applied this method to

prove non-invertibility of an infinite family of twist spins of torus knots and their stabilizations. A similar result has been obtained for 2-bridge knots by Iwakiri[17]. From the generalized invariant, we have:

Theorem 8.1 [4] *For any positive integer k , the $2k$ -twist spun of all the 3-colorable knots in the table up to 9 crossings excluding 8_{20} , as well as their stabilized surfaces of any genus, are non-invertible.*

Some of the computations that yield this result are presented in Table 1.

Minimal triple point numbers of projections of knotted surfaces.

Classical knot tables have played a pivotal role in the history of classical knot theory. Knot tables are organized according to (minimal) crossing numbers. An analogue is the minimal number of triple points among all generic projections of a knotted surface, called the *triple point number* $\mathcal{T}(F)$ of a knotted surface F .

Progress has been made in [33, 34, 37] about triple point numbers, but there were no examples of knotted spheres whose triple point numbers were concretely determined until a breakthrough was given in [35, 36], in which the following were proved:

$$\begin{aligned}\mathcal{T} (\text{2-twist spun trefoil}) &= 4, \\ \mathcal{T} (\text{3-twist spun trefoil}) &= 6.\end{aligned}$$

Since then further applications of quandle colorings and the cocycle invariants to the triple point numbers have been obtained ([15], for example).

Remark 8.2 These results can be interpreted as a pseudo-norm on quandle homology. Specifically, the minimum number of generators with which a non-trivial homology class is represented is regarded as a pseudo-norm and gives a lower bound for the triple point numbers.

Generalized cocycle invariants often give higher lower bounds. For example, it can be used to reprove:

Theorem 8.3 ([19], see also [24]) *For any positive integer n , there exists a knotted surface K such that $\mathcal{T}(K) > n$.*

In fact, the theorem is verified by specific examples. With R_3 and a certain cocycle, it is proved that if a knotted surface K has a non-trivial cocycle invariant, then for any positive integer n , there is a positive integer k such that $\mathcal{T}(\tau^{2k}(K)) \geq n$.

Ribbon concordance of knotted surfaces. Let F_0 and F_1 be connected knotted surfaces of the same genus. We say that F_1 is *ribbon concordant* to F_0 if there is a concordance C in $\mathbb{R}^4 \times [0, 1]$ between $F_1 \subset \mathbb{R}^4 \times \{1\}$ and $F_0 \subset \mathbb{R}^4 \times \{0\}$ such that the restriction to C of the projection $\mathbb{R}^4 \times [0, 1] \rightarrow [0, 1]$ is a Morse function with critical points of index 0 and 1 only. We write $F_1 \geq F_0$. Note that if $F_1 \geq F_0$, then there is a set of n 1-handles on a split union of F_0 and n trivial sphere, for some $n \geq 0$, such that F_1 is obtained by surgeries along these handles (Fig. 4).

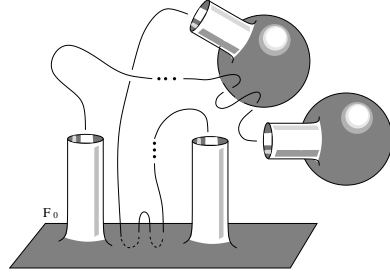


Figure 4: Ribbon concordance

The notion of ribbon concordance was originally introduced by Gordon [14] for classical knots in \mathbb{R}^3 , and there are several studies found in [13, 26, 27, 38], for example.

Given knotted surfaces F_0 and F_1 , it is natural to ask whether F_1 is ribbon concordant to F_0 . Cochran [7] gave a necessary condition for a sphere-knot F to be ribbon in terms the knot group $\pi_1(\mathbb{R}^4 \setminus F)$. In [6], new necessary conditions were given for a pair of knotted surfaces to be ribbon concordant by using quandle cocycle invariants.

The cocycle invariant $\Phi_\theta(F)$ is regarded as a multi-set of elements in the coefficient group A of the cohomology where repetitions of the same element are allowed. For two multi-sets A' and A'' of A , we use the notation $A' \overset{m}{\subset} A''$ if for any $a \in A'$ it holds that $a \in A''$. In other words, $A' \overset{m}{\subset} A''$ if and only if $\tilde{A}' \subset \tilde{A}''$ where \tilde{A}' and \tilde{A}'' are the subsets of A obtained from A' and A'' by eliminating the multiplicity of elements, respectively.

Theorem 8.4 [6] *If $F_1 \geq F_0$, then $\Phi_\theta(F_1) \overset{m}{\subset} \Phi_\theta(F_0)$.*

By Theorem 8.4, we give many examples of pairs of knotted surfaces such that one is *not* ribbon concordant to another.

To generalize this result to surfaces without triple points, a new cocycle invariants defined on $H_1(F)$ for a surface F was constructed in [6] as well.

Remark 8.5 Kawauchi points out that the linking signature of a certain family of surfaces is invariant under ribbon concordance. This result has not appeared in any paper, but can be obtained as a corollary of [23].

9 Module invariants and twisted Alexander invariants

A braid word w (of k -strings), or a k -braid word, is a product of standard generators $\sigma_1, \dots, \sigma_{k-1}$ of the braid group \mathcal{B}_k of k -strings and their inverses. A braid word w represents an element $[w]$ of the braid group \mathcal{B}_k . Geometrically, w is represented by a diagram in a rectangular box with k end points at the top, and k end points at the bottom, where the strings go down monotonically. Each generator or its inverse is represented by a crossing in a diagram. We use the same letter w for a choice of such a diagram. Let \hat{w} denote the closure of the diagram w . Quandle colorings of w are defined in exactly the same manner as in the case of knots. However, the quandle elements at the top and the bottom of a diagram of w do not necessarily coincide. But when we consider a coloring of the link \hat{w} , the quandle elements at the top and the bottom of a diagram of w do coincide.

Let X be a quandle. Let $\gamma_1, \dots, \gamma_k$ be the bottom arcs of w . For a given vector $\vec{x} = (x_1, \dots, x_k) \in X^k$, assign these elements x_1, \dots, x_k on $\gamma_1, \dots, \gamma_k$ as their colors, respectively. Then from the definition, a coloring \mathcal{C} of w by X is uniquely determined such that $\mathcal{C}(\gamma_i) = x_i$, $i = 1, \dots, k$. We call such a coloring \mathcal{C} the *coloring induced from \vec{x}* . Let $\delta_1, \dots, \delta_k$ be the arcs at the top. Let $\vec{y} = (y_1, \dots, y_k) = (\mathcal{C}(\delta_1), \dots, \mathcal{C}(\delta_k)) \in X^k$ be the colors assigned to the top arcs. Denote this situation by a left action, $\vec{y} = w \cdot \vec{x}$. The colors \vec{x} and \vec{y} are called *bottom* and *top colors* or *color vectors*, respectively. See Fig. 5.

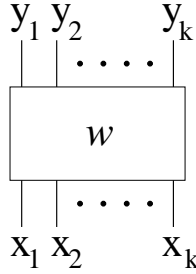


Figure 5: A quandle coloring of a braid word w

Let X be a quandle and G be a quandle module. For $\alpha = \eta + \tau$ which acts on $(a, b) \in G^2$ by $\alpha_{x,y}(a, b) = \eta_{x,y}(a) + \tau_{x,y}(b)$ for any $(x, y) \in X^2$. Let

$\tilde{X} = G \times_{\alpha} X$ be the quandle defined by $(a, x) * (b, y) = (\alpha_{x,y}(a, b), x * y)$ (which is called the dynamical extension). If $\vec{r} = ((a_1, x_1), \dots, (a_k, x_k))$ and $\vec{s} = ((b_1, y_1), \dots, (b_k, y_k)) \in \tilde{X}^k$ are bottom and top colors of $w \in \mathcal{B}_k$ by \tilde{X} , respectively, then we write this situation by $\vec{b} = M(w, \vec{x}) \cdot \vec{a}$, where $\vec{a} = (a_1, \dots, a_k)$, $\vec{b} = (b_1, \dots, b_k) \in G^k$. Thus $M(w, \vec{x})$ represents a map $M(w, \vec{x}) : G^k \rightarrow G^k$.

Lemma 9.1 [4] *If $[w] = [w'] \in \mathcal{B}_k$, then $M(w, \vec{x}) = M(w', \vec{x}) : G^k \rightarrow G^k$.*

We call the map $M(-, \vec{x}) : \mathcal{B}_k \rightarrow \text{Map}(G^k, G^k)$ a *colored representation*.

Theorem 9.2 [4] *Let L be a link represented as a closed braid \hat{w} , where w is a k -braid word, and $\text{Col}_X(L)$ be the set of colorings of L by a quandle X . For $\mathcal{C} \in \text{Col}_X(L)$, let \vec{x} be the color vector of bottom strings of w that is the restriction of \mathcal{C} . Then the family*

$$\mathcal{M}(X, \alpha ; L) = \{G^k / \text{Im}(M(w, \vec{x}) - I)\}_{\mathcal{C} \in \text{Col}_X(L)}$$

of isomorphism classes of modules presented by the maps $(M(w, \vec{x}) - I)$, where I denotes the identity, is independent of choice of w that represents L as its closed braid, and thus defines a link invariant.

This theorem implies that the following is well-defined.

Definition 9.3 [4] *The family of modules $\mathcal{M}(X, \alpha ; L) = \{G^k / \text{Im}(M(w, \vec{x}) - I)\}_{\mathcal{C} \in \text{Col}_X(L)}$ is called the *quandle module invariant*.*

This invariant is related to the twisted Alexander invariant [25, 29, 40]. Let $\pi = \pi_1(S^3 \setminus K) = \langle x_1, \dots, x_s | r_1, \dots, r_k \rangle$ be a Wirtinger presentation of a classical knot K , so that relations are of the form $x_{\ell} = x_s x_j x_s^{-1}$. Let $V = \mathbb{Z}^n$ and $\rho : \pi_1 \rightarrow GL(V)$ be a representation, and $\epsilon : \pi_1 \rightarrow \mathbb{Z}$ be the abelianization. Finally, let $\mathbb{Z}[F_s]$ be the free group ring generated by x_1, \dots, x_s and $\chi : \mathbb{Z}[F_s] \rightarrow \mathbb{Z}[\pi] \rightarrow M_n(\mathbb{Z}[t, t^{-1}])$ be the map that is determined by $\rho \otimes \epsilon : \mathbb{Z}[\pi] \rightarrow M_n(\mathbb{Z}[t, t^{-1}])$ that sends γ to $t^{\epsilon(\gamma)} \rho(\gamma)$. Then the module with the presentation $(sn \times kn)$ -matrix $\left[\chi \left(\frac{\partial r_i}{\partial x_j} \right) \right]$ is used to define the twisted Alexander invariant.

On the other hand, we can define a quandle module structure by using the action of G_X on $(\mathbb{Z}[t, t^{-1}])^n$ defined by $\rho \otimes \epsilon : \mathbb{Z}[\pi] \rightarrow M_n(\mathbb{Z}[t, t^{-1}])$. Then the quandle module invariant has a similar presentation matrix as above. Therefore a detailed analysis of the roles played by the matrix $\left[\chi \left(\frac{\partial r_i}{\partial x_j} \right) \right]$ will yield connections among these subjects.

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